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2002 J. Phys. A: Math. Gen. 35 6659

(http://iopscience.iop.org/0305-4470/35/31/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6659–6673 PII: S0305-4470(02)35300-9

On scattering systems related to the $SO(2, 1)$ group: II

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Received 28 March 2002, in final form 12 June 2002
Published 26 July 2002
Online at stacks.iop.org/JPhysA/35/6659

Abstract
We study a class of solvable Hamiltonians H which have strongly broken potential group structure. However, the scattering matrices of the systems under consideration are also related to the intertwining operator of $SO(2, 1)$.

PACS numbers: 03.65.Fd, 02.20.Sv, 03.65.Nk

1. Introduction

Since the advent of quantum mechanics it has been obvious that algebraic structures play a fundamental role in quantum theory. Indeed, the first quantum study of the hydrogen atom [1] was based upon the algebra generated by angular momentum and the Runge–Lenz vector. The prescription of this invariance algebra, which is isomorphic to the $so(4)$ Lie algebra, allows a determination of the energy spectrum of the bound states of the hydrogen atom. Since then, invariance algebras have been determined for many quantum mechanical systems. This is a situation in which the Hamiltonian H of the system is expressed in terms of the Casimir operator C of some algebra $\mathfrak{g}$, i.e. $H = f(C)$. For example, in the hydrogen bound-state problem, $H = \alpha/(C - 1)$, where C is the second-order Casimir operator of $so(4)$.

Since the work of Zwanzinger [2] it has become clear that an algebraic approach can be successfully applied to the solution of scattering problems. Important results have been obtained in this way by the Yale group and others [3–11]. To extend the algebraic approach to other scattering systems, another kind of algebraic structure, the so-called potential algebra, was suggested in [3]; the Hamiltonians of the one-dimensional systems are related to the Casimir operator C of the noncompact algebra $\mathfrak{g}$ as

$$H = f(C)|_{\mathcal{H}} \tag{1.1}$$

where $\mathcal{H}$ are one-dimensional subspaces of the carrier space. (As usual, H is linear on C and $\mathcal{H}$ are the eigensubspaces of the compact generators). Hence, the potential algebra describes fixed energy states of a family of one-dimensional systems with different potential strength. The next step was taken by Frank and Wolf [4] (see also [5]), who utilized the potential algebra $so(2, 1)$ to construct the S matrix for the Pöschl–Teller potential [12]. However, the method suggested there used an explicit coordinate realization. Subsequently, following the ideas of [4] Alhassid *et al* [6] suggested a purely algebraic description of the S-matrix associated with $so(2, 1)$ algebra. It appears that [6–9] knowledge of the interrelation between an algebra, which describes the dynamics of the scattering system, and an Euclidean algebra, which describes the asymptotic properties of the system, allows in principle, pure algebraic calculation of S-matrices. This technique, which is called the Euclidean connection, essentially uses the theory of group deformations [13].

At this point we mention that the approach presented in [3–7] is similar to the Olshanetsky–Perelomov approach [14] to quantum integrable systems related to Lie algebras (where the Hamiltonians of the systems are described in terms of the radial part of the Casimir operator). Therefore, one may, in principle, extend the method of algebraic evaluation of the scattering matrix to many-body scattering problems related to (higher real-rank) Lie algebras. Unfortunately the theory of group deformations has not yet been developed as far as one would wish; there exist a number of results about most degenerate representations of some higher real-rank algebras (see, e.g., [9] and the references cited therein). Therefore, it is rather difficult to derive the S-matrix for the many-body scattering problems using the above-mentioned method.

It should be noted that the potential group approach initiated in [3] is a rediscovery of a technique attributable to Ghirardi [15] (see equation (3.2) of [15]). In that paper Ghirardi also proposed an algebraic method in which the Hamiltonians H of the systems are related to the Casimir operator C of $so(2, 1)$ as

$$Q(x)(H - E) = [C - j(j + 1)]|_{\mathcal{H}} \tag{1.2}$$

where j specifies the discrete series representations of $so(2, 1)$ and $\mathcal{H}$ is an eigensubspace of the compact generator. (Observe, at $Q(x) = \text{const}$ we consider models with potential group structure.)

Since knowledge of invariance algebra is sufficient for solving the bound-state problems, it is quite suggestive to ask whether or not one can use information on the invariance algebra directly to determine the scattering matrices completely. The answer is in the affirmative [16]. It has been discovered that S-matrices for systems under consideration are related to the intertwining operators between the Weyl equivalent principal series representations of the invariance algebra $\mathfrak{g}$. Namely, the S-matrix is constrained to satisfy

$$S\, \mathrm{d}U^{\chi}(X) = \mathrm{d}U^{\widetilde{\chi}}(X)S \qquad \text{for all } X \in \mathfrak{g} \tag{1.3}$$

or

$$SU^{\chi}(g) = U^{\widetilde{\chi}}(g)S \qquad \text{for all } g \in G \tag{1.4}$$

where $\mathrm{d}U^{\chi}$ and $\mathrm{d}U^{\widetilde{\chi}}$ are the Weyl equivalent principal series representations of $\mathfrak{g}$ while U^{χ} and $U^{\widetilde{\chi}}$ are the corresponding representations of the group G with $\mathfrak{g}$. Equations (1.3) and (1.4) have great restrictive power, determining the S-matrix up to a constant. Thus, one can in principle evaluate the S-matrix from (1.3) or (1.4) without writing a Schrödinger equation, or wavefunctions, or mentioning the concepts of space and time. We note that the operator S with property (1.3) (or (1.4)) is called an intertwining operator [17] between $\mathrm{d}U^{\chi}$ and $\mathrm{d}U^{\widetilde{\chi}}$ (U^{χ} and $U^{\widetilde{\chi}}$).

Moreover, it follows from equation (1.3) or (1.4) that if the matrix of the representation operator is diagonal in some basis then the matrix of the intertwining operator is also diagonal. This fact leads to the suggestion that there might exist a class of one-dimensional potentials for which the scattering matrix is determined by diagonal elements of the intertwining operator. This is exactly what happens in the algebraic approaches presented in [3–7, 10, 13]. Thus, the number of subgroup chains provided by the representation theory necessarily corresponds to the number of classes of quantum systems. Therefore the problem of classification of all one-dimensional systems related to group G may be reduced to the more tractable problem of enumeration of all subgroup chains of G. Moreover, one can use the well-developed theory of intertwining operators for semi-simple Lie groups [18] to obtain a stringent restriction upon the structure of the scattering matrices for many-body systems associated with semi-simple Lie algebras, or even to determine it completely [19].

In a previous paper [20], we discussed the scattering problems related to $SO(2,1)$. It has been shown that the scattering problem can be completely solved within the framework of group theory, without explicit knowledge of the interaction potentials. It has also been shown that according to $SO(2,1) \supset SO(2)$, $SO(2,1) \supset SO(1,1)$ and $SO(2,1) \supset E(1)$ subgroup reductions one has three classes of one-dimensional scattering problems related to $SO(2,1)$. The S-matrix for such systems is given by

(i) Class 1 (related to $SO(2,1) \supset SO(2)$ reduction)

$$S_m = c(\rho)\frac{\Gamma\left(\frac{1}{2} - \mathrm{i}\rho + m\right)}{\Gamma\left(\frac{1}{2} + \mathrm{i}\rho + m\right)} \tag{1.5}$$

(ii) Class 2 (related to $SO(2,1) \supset SO(1,1)$ reduction)

$$S_\nu = \begin{pmatrix} R_\nu & T_\nu \\ T_\nu & R_\nu \end{pmatrix} \tag{1.6}$$

where

$$R_\nu = c(\rho)\cosh(\pi\nu)\Gamma\left(\frac{1}{2} - \mathrm{i}\rho + \mathrm{i}\nu\right)\Gamma\left(\frac{1}{2} - \mathrm{i}\rho - \mathrm{i}\nu\right)$$
$$T_\nu = -\mathrm{i}c(\rho)\frac{1}{\pi}\sinh(\pi\rho)\Gamma\left(\frac{1}{2} - \mathrm{i}\rho + \mathrm{i}\nu\right)\Gamma\left(\frac{1}{2} - \mathrm{i}\rho - \mathrm{i}\nu\right)$$

(iii) Class 3 (related to $SO(2,1) \supset E(1)$ reduction)

$$S_\lambda = c(\rho)|\lambda|^{-2\mathrm{i}\rho} \tag{1.7}$$

where $c(\rho)$ is an arbitrary phase factor; ρ, m, ν and λ specify the irreducible representations of $SO(2,1)$, $SO(2)$, $SO(1,1)$ and $E(1)$, respectively.

Following the ideas of Ghirardi one can propose one-dimensional scattering systems whose Hamiltonians are related to the Casimir operator C of $SO(2,1)$ as

$$Q(x)(H - E) = [C - j(j+1)]|_{\mathcal{H}} \qquad j = -\tfrac{1}{2} - \mathrm{i}\rho \tag{1.8}$$

where $\mathcal{H}$ are the one-dimensional subspaces of the carrier space occurring in the above-mentioned subgroup reductions. It is clear that the scattering matrices for such systems are also given by formulae (1.5)–(1.7). However, in this case all parameters ρ^2, m^2, ν^2 and λ^2 are linear functions of the energy E. (It should be stressed that for models with potential group structure the quantum number ρ is related to the energy, while m, ν and λ are taken to be independent of energy; the energy dependence of ρ is determined by the relation connecting H and C.)

The question that arises, then, is: what are the interaction potentials for which relation (1.8) holds? In a previous paper we gave the simple example of how the problem for systems with the $SO(2,1)$ potential group structure can be solved within the framework of group theory. Here we show that the solution of the problem in the case of models with algebraic structure proposed in (1.8) is also possible within this framework.

2. Solvable potentials related to $SO(2,1)$

We want to deal with single particle scattering by one-dimensional potentials related to the principal series of $SO(2,1)$ in the sense that relation (1.8) holds. To this end, a few facts from the representation theory of the $SO(2,1)$ are useful.

Let $R^{2,1}$ be a three-dimensional pseudo-Euclidean space with bilinear form

$$[\xi,\eta] = \xi_0\eta_0 - \xi_1\eta_1 - \xi_2\eta_2. \tag{2.1}$$

By $SO(2,1)$ we denote the connected component of the group of linear transformations of $R^{2,1}$ preserving the form (2.1). We consider $SO(2,1)$ as acting on $R^{2,1}$ on the right. In accordance with this we shall write the vector in the row form $\xi = (\xi_0, \xi_1, \xi_2)$. The unitary irreducible representations (UIRs) of $SO(2,1)$ are known to form three series [21]: principal, supplementary and discrete. Since we want to deal with particle scattering, the relevant unitary representations will be the principal series and we restrict the discussion to it.

The principal series of $SO(2,1)$ are labelled by ρ, with $0 \leqslant \rho < \infty$. The representations specified by labels ρ and $-\rho$ are Weyl equivalent. The generators of the representation of the Lie algebra of $SO(2,1)$ associated with the principal series are denoted by J_i, $i = 0, 1, 2$, where J_0 is the generator corresponding to the rotations in the 1–2 plane

$$g_0(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \tag{2.2}$$

while J_1 and J_2 are the generators corresponding to the pure Lorentz transformations along the 1 and 2 axes, respectively

$$g_1(t) = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_2(t) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}. \tag{2.3}$$

J_i are the Hermitian operators and satisfy the commutation relations

$$[J_1, J_2]_- = -\mathrm{i}J_0 \qquad [J_2, J_0]_- = \mathrm{i}J_1 \qquad [J_0, J_1]_- = \mathrm{i}J_2. \tag{2.4}$$

The operator J_0 is elliptic, while J_1 and J_2 are hyperbolic. The Casimir operator

$$C = J_0^2 - J_1^2 - J_2^2 \tag{2.5}$$

is identically a multiple of the unit $C = -\frac{1}{4} - \rho^2$.

As is well known, the group $SO(2,1)$ has three subgroups $SO(2)$, $SO(1,1)$ and $E(1)$, where $E(1)$ (being isomorphic to the Euclidean group in one-dimension) consists of matrices of the form

$$n(t) = \begin{pmatrix} 1 + t^2/2 & -t^2/2 & t \\ t^2/2 & 1 - t^2/2 & t \\ t & -t & 1 \end{pmatrix}. \tag{2.6}$$

Hence, we are interested in the principal series of $SO(2,1)$ in $SO(2)$, $SO(1,1)$ and $E(1)$ bases in which the operators J_0, J_1 and $N = J_0 - J_2$ are diagonal, respectively.

We now return to our main theme. We want to construct the Hamiltonians for which relation (1.8) holds. The key to their construction lies in the observation that the Schrödinger energy eigenvalue equation for such systems is nothing but the condition imposed on the carrier space of $SO(2,1)$ to be irreducible. Thus in order to find the Hamiltonians for the systems under consideration we should look for a reducible representation of $SO(2,1)$ containing the UIR of the principal series.

Let us consider a quasiregular representation of $SO(2,1)$ induced by a one-dimensional identity representation of $SO(2)$ [21]. We note that this representation is decomposed into the direct integral of principal series representations. Hence, the principal series representations can be realized as a subrepresentation of the quasiregular one.

The quasiregular representation can be realized in the Hilbert space $L^2(\Xi, \mathrm{d}\mu)$ of square-integrable functions on an upper sheet of hyperboloid $\Xi = SO(2,1)/SO(2)$

$$\xi_0^2 - \xi_1^2 - \xi_2^2 = 1 \qquad \xi_0 > 0. \tag{2.7}$$

Generally, for the construction of the quasiregular representation one can use the carrier space $L^2(\Xi, \mathrm{d}\mu)$ with any quasi-invariant measure $\mathrm{d}\mu(\xi)$ on Ξ. The representation is given by [21]

$$T(g)f(\xi) = (\mathrm{d}\mu(\xi g)/\mathrm{d}\mu(\xi))^{1/2} f(\xi g) \tag{2.8}$$

with inner product

$$(f, f') = \int \overline{f(\xi)} f'(\xi)\, \mathrm{d}\mu(\xi) \tag{2.9}$$

where $\mathrm{d}\mu(\xi g)/\mathrm{d}\mu(\xi)$ is the Radon–Nikodym derivative. The representations with different measure are unitarily equivalent.

In the case of $\mathrm{d}\mu(\xi) = \mathrm{d}\xi$, where $\mathrm{d}\xi \equiv \mathrm{d}\xi_1\, \mathrm{d}\xi_2/\xi_0$ is an invariant measure on Ξ, the Radon–Nikodym derivative equals 1 and the representation, called $\check{T}$, has the simple form

$$\check{T}(g)\check{f}(\xi) = \check{f}(\xi g) \tag{2.10}$$

with inner product

$$(\check{f}, \check{f}') = \int \overline{\check{f}(\xi)} \check{f}'(\xi)\, \mathrm{d}\xi. \tag{2.11}$$

We are now prepared to construct the principal series of $SO(2,1)$ as a subrepresentation of T. To do this, we require the representation space to be irreducible. Such a restriction is obtained if all functions f are eigenfunctions of the Casimir operator $C = J_0^2 - J_1^2 - J_2^2$ of T

$$Cf = j(j+1)f \qquad j = -\tfrac{1}{2} - \mathrm{i}\rho \tag{2.12}$$

where J_k are infinitesimal operators of the representation (2.8)

$$J_k = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} T(g_k(t))|_{t=0} \qquad k = 0, 1, 2. \tag{2.13}$$

Next, imposing the reduction condition, one can choose a different basis in the carrier space.

As mentioned above, the quasiregular representations with different measure are unitarily equivalent. Although the representations with different measure are mathematically equivalent, they may be related to different physical problems. For this reason, we shall consider the quasiregular representation with different measure.

2.1. A class of potentials related to $SO(2,1) \supset SO(2)$

According to this we want $\mathrm{d}\mu$ to be invariant under $SO(2)$. We can, without loss of generality, put $\mathrm{d}\mu(\xi) = v(\xi_0)\, \mathrm{d}\xi$ where $\mathrm{d}\xi$ is the invariant measure on Ξ. The requirement that the measure is quasi-invariant implies only the condition

$$v(\xi_0) \geqslant 0. \tag{2.14}$$

Such defined quasiregular representation, called T, of course, is unitarily equivalent to $\check{T}$. The unitary mapping W which realizes the equivalence is given by

$$W : f \longrightarrow \check{f} = v^{1/2} f. \tag{2.15}$$

In this case, the generators and the Casimir operator, denoted as J_1, J_2, J_0 and C, are given by

$$J_0 = \mathrm{i}\xi_2 \frac{\partial}{\partial \xi_1} - \mathrm{i}\xi_1 \frac{\partial}{\partial \xi_2} \qquad J_1 = \mathrm{i}\xi_0 \frac{\partial}{\partial \xi_1} + \frac{\mathrm{i}\xi_0}{2v} \frac{\partial v}{\partial \xi_1} \qquad J_2 = \mathrm{i}\xi_0 \frac{\partial}{\partial \xi_2} + \frac{\mathrm{i}\xi_0 \xi_2}{2\xi_1 v} \frac{\partial v}{\partial \xi_1}$$

and

$$C = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \left(\frac{\xi_0^2}{\xi_1 v} \frac{\partial v}{\partial \xi_1} + 1 + \Lambda \right) \Lambda + \frac{\xi_0^2 \left(\xi_0^2 - 1\right)}{4v} \left[\frac{1}{\xi_1 \xi_2} \frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} - \frac{1}{\xi_1^2 v} \left(\frac{\partial v}{\partial \xi_1} \right)^2 - \frac{2\left(1 - 3\xi_0^2\right)}{\xi_1 \xi_0^2 \left(\xi_0^2 - 1\right)} \frac{\partial v}{\partial \xi_1} \right]$$

with

$$\Lambda = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}$$

where we have used

$$\frac{\partial v}{\partial \xi_2} = \frac{\xi_2}{\xi_1} \frac{\partial v}{\partial \xi_1}. \tag{2.16}$$

(We are taking ξ_1 and ξ_2 as the independent variables on Ξ.)

Since J_0 is sought to be diagonal, we introduce in the place of ξ_1, ξ_2 the variables x, φ via

$$\xi_1 = \frac{2\sqrt{z(x)}}{1 - z(x)} \cos\varphi \qquad \xi_2 = \frac{2\sqrt{z(x)}}{1 - z(x)} \sin\varphi \tag{2.17}$$

with $0 \leqslant \varphi < 2\pi$, $0 \leqslant x < \infty$, where z is a differentiable function on R^+ with values in $[0, 1]$. So J_0 becomes the operator $-\mathrm{i}\frac{\partial}{\partial\varphi}$, while

$$C = \frac{z(1-z)^2}{\dot{z}^2} \left\{ \frac{\partial^2}{\partial x^2} + \left(\frac{\dot{v}}{v} - \frac{\ddot{z}}{\dot{z}} + \frac{\dot{z}}{z} \right) \frac{\partial}{\partial x} + \frac{\dot{z}^2}{4z^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{2v} \left[\ddot{v} - \frac{\dot{v}^2}{2v} - \left(\frac{\ddot{z}}{\dot{z}} - \frac{\dot{z}}{z} \right) \dot{v} \right] \right\} \tag{2.18}$$

where dots represent derivatives with respect to x, i.e. $\dot{z} = \mathrm{d}z/\mathrm{d}x$, $\ddot{z} = \mathrm{d}^2 z/\mathrm{d}x^2$, etc. In order to eliminate the term containing the first derivative we require

$$\frac{\dot{v}}{v} - \frac{\ddot{z}}{\dot{z}} + \frac{\dot{z}}{z} = 0. \tag{2.19}$$

Hence we have that up to a common factor

$$v = \dot{z} z^{-1}. \tag{2.20}$$

Since v must be positive we require that $\dot{z} > 0$. Substituting equation (2.20) into equation (2.18), one gets

$$C = \frac{z(1-z)^2}{\dot{z}^2} \left[\frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{\dddot{z}}{\dot{z}} - \frac{3}{4} \left(\frac{\ddot{z}}{\dot{z}} \right)^2 + \frac{\dot{z}^2}{4z^2} \left(1 + \frac{\partial^2}{\partial \varphi^2} \right) \right]. \tag{2.21}$$

Thus, the principal series of $SO(2, 1)$ in $SO(2)$ basis can be realized in the Hilbert space spanned by eigenfunctions of C and J_0

$$C f_m^{(1)} = \left(-\tfrac{1}{4} - \rho^2 \right) f_m^{(1)} \qquad J_0 f_m^{(1)} = m f_m^{(1)} \tag{2.22}$$

where $J_0 = -\mathrm{i}\frac{\partial}{\partial\varphi}$ and C is given by (2.21). Let $\mathcal{H}_m$ be a one-dimensional subspace spanned by $f_m^{(1)}$ with given m. Then the Casimir operator restricted to $\mathcal{H}_m$ becomes a differential operator in x alone; it is found that

$$C_m = \frac{z(1-z)^2}{\dot{z}^2} \left[\frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{\dddot{z}}{\dot{z}} - \frac{3}{4} \left(\frac{\ddot{z}}{\dot{z}} \right)^2 + \frac{\dot{z}^2 (1 - m^2)}{4z^2} \right] \tag{2.23}$$

where C_m denotes the restriction of C to $\mathcal{H}_m$. We note that the operator C_m is the Schrödinger type if

$$\frac{z(1-z)^2}{\dot{z}^2} = 1. \tag{2.24}$$

The solution to this equation is given by

$$z = \tanh^2 \frac{x}{2}. \tag{2.25}$$

With this z, the operator C_m is related to the Pöschl–Teller Hamiltonian

$$H_m = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m^2 - \frac{1}{4}}{\sinh^2 x} \tag{2.26}$$

as $H_m = -\left(C_m + \frac{1}{4}\right)$. (We are using units with $2M = \hbar = 1$.) Thus, the Pöschl–Teller Hamiltonian (2.26) has $SO(2,1)$ as the potential group; the scattering states that have the same energy but belong to different potential strengths are related to the UIR of the principal series of $SO(2,1)$. However, for a class of Hamiltonians relation (1.8) can be satisfied by the proper choice of m^2 and ρ^2 as a function of the energy. It is not difficult to see that for Hamiltonians

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{h_0 z^2 + (h_1 - 2h_0)z + h_0 + 1}{R} + \frac{z^2(1-z)^2}{R^2} \times \left(c_0 + \frac{2(c_1 - c_0)z + 2c_0 - c_1}{z(z-1)} - \frac{5\Delta}{4R}\right) \tag{2.27}$$

where $\Delta = c_1^2 - 4c_0c_1$, $R(z) = c_0z^2 + (c_1 - 2c_0)z + c_0$ and $z\,(x)$ satisfies

$$\dot{z} = \frac{2z(1-z)}{\sqrt{R(z)}}$$

the following relation holds

$$\left(C_m + \rho^2 + \frac{1}{4}\right) = -\frac{z(1-z)^2}{\dot{z}^2}[H - E] \tag{2.28}$$

provided

$$1 - m^2 = c_0E - h_0 \qquad 1 + 4\rho^2 = c_1E - h_1. \tag{2.29}$$

As a consequence, one finds exactly solvable Hamiltonians (2.27) which have a 'broken symmetry' in the sense that $H \neq f(C)|_{\mathcal{H}_m}$. However, the Hamiltonians (2.27) have another kind of algebraic structure. It follows from (2.28) that

$$\left(C + \rho^2 + \tfrac{1}{4}\right)\Big|_{\mathcal{H}_m} = Q(x)(H - E)$$

with m and ρ given by (2.29) and $Q(x) = -\frac{z(1-z)^2}{\dot{z}^2}$.

The class of Hamiltonians (2.27) contains as a particular case the Pöschl–Teller Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{h_0 + \frac{3}{4}}{\sinh^2 x}$$

which was already known to possess $SO(2,1)$ as the potential group. (In this case $c_0 = 0$, $c_1 = 4$ and $h_1 = -1$.) It is also worth mentioning that the interaction potentials given in (2.27) belong to a class of Natanzon hypergeometric potentials [17] which depends on six parameters f, h_0, h_1, a, c_0 and c_1 (see also [10, 11, 23]). For the potentials given in (2.27) $a = c_0$ and $f = h_0$. (In (2.27) we closely following the notation of [17].)

Thus, the wavefunctions of the restricted class of Natanzon hypergeometric potentials (which depends on four parameters) are related to the basis functions $f_m^{(1)}(\xi)$, while the S-matrix is determined by diagonal elements (1.5) of the intertwining matrix. It follows from (2.15) and (A.7) that

$$f_m^{(1)}(\xi) = \left(\frac{z}{\dot z}\right)^{\frac{1}{2}} \int_0^{2\pi} \left(\frac{1+z}{1-z} - \frac{2\sqrt{z}}{1-z}\cos(\theta-\varphi)\right)^{-1-j} \mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{d}\theta \qquad (2.30)$$

where $j = -\frac{1}{2} - \mathrm{i}\rho$. Hence for the wavefunctions of the restricted Natanzon potentials given in (2.27) we have

$$\Psi(x) \propto (\dot z)^{-\frac{1}{2}} z^{\frac{1+m}{2}} (1-z)^{1+j}\, {}_2F_1(1+j, 1+j+m; 1+m; z)$$

where ${}_2F_1$ are the standard hypergeometric functions.

2.2. *A class of potentials related to $SO(2,1) \supset SO(1,1)$*

Now we require the quasi-invariant measure $\mathrm{d}\mu$ to be invariant under the transformations $g_1(t) \in SO(1,1)$ (see equation (2.3)). According to this, we put $\mathrm{d}\mu = v(\xi_2)\,\mathrm{d}\xi$. (For the sake of simplicity, we will denote the generators and the Casimir operator by J_i and C, respectively.) Then

$$J_0 = \mathrm{i}\xi_2\frac{\partial}{\partial\xi_1} - \mathrm{i}\xi_1\left(\frac{1}{2v}\frac{\partial v}{\partial\xi_2} + \frac{\partial}{\partial\xi_2}\right) \qquad J_1 = \mathrm{i}\xi_0\frac{\partial}{\partial\xi_1} \qquad J_2 = \mathrm{i}\xi_0\frac{\partial}{\partial\xi_2} + \frac{\mathrm{i}\xi_0}{2v}\frac{\partial v}{\partial\xi_2}$$

and

$$C = \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} + \left(\frac{\xi_2}{v}\frac{\partial v}{\partial\xi_2} + 1 + \Lambda\right)\Lambda + \frac{1}{v}\frac{\partial v}{\partial\xi_2}\frac{\partial}{\partial\xi_2} + \frac{\left(1+\xi_2^2\right)}{2v}\left[\frac{\partial^2 v}{\partial\xi_2^2} - \frac{1}{2v}\left(\frac{\partial v}{\partial\xi_2}\right)^2 + \frac{2\xi_2}{1+\xi_2^2}\frac{\partial v}{\partial\xi_2}\right]. \qquad (2.31)$$

In place of ξ_1, ξ_2 let us introduce the new variables x, β by

$$\xi_1 = \frac{\sinh\beta}{\sqrt{1-z^2(x)}} \qquad \xi_2 = \frac{z(x)}{\sqrt{1-z^2(x)}} \qquad (2.32)$$

with $-\infty < \beta < \infty$ and $-\infty < x < \infty$, where now z is a differentiable function on R with values in $[-1, 1]$. Then J_1 becomes the operator $\mathrm{i}\frac{\partial}{\partial\beta}$, while

$$C = \frac{(1-z^2)^2}{\dot z^2}\left\{\frac{\partial^2}{\partial x^2} + \left(\frac{\dot v}{v} - \frac{\ddot z}{\dot z} - \frac{\dot z z}{1-z^2}\right)\frac{\partial}{\partial x} + \frac{\dot z^2}{1-z^2}\frac{\partial^2}{\partial\beta^2} + \frac{1}{2v}\left[\ddot v - \frac{\dot v^2}{2v} - \left(\frac{\ddot z}{\dot z} + \frac{\dot z z}{1-z^2}\right)\dot v\right]\right\}. \qquad (2.33)$$

We now require

$$\frac{\dot v}{v} - \frac{\ddot z}{\dot z} - \frac{\dot z z}{1-z^2} = 0$$

which yields

$$v = \dot z(1-z^2)^{-1/2}. \qquad (2.34)$$

Since v must be positive we require that $\dot z > 0$. Putting equation (2.34) into equation (2.33), one gets

$$C = \frac{(1-z^2)^2}{\dot z^2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\dddot z}{\dot z} - \frac{3}{4}\left(\frac{\ddot z}{\dot z}\right)^2 + \frac{\dot z^2(2+z^2)}{4(1-z^2)^2} + \frac{\dot z^2}{1-z^2}\frac{\partial^2}{\partial\beta^2}\right]. \qquad (2.35)$$

The basis functions corresponding to the considered reduction are the eigenfunctions of the set of operators C and J_1

$$Cf^{(2)}_{\nu\tau} = \left(-\tfrac{1}{4} - \rho^2\right) f^{(2)}_{\nu\tau} \qquad J_1 f^{(2)}_{\nu\tau} = \nu f^{(2)}_{\nu\tau}$$

where $J_1 = \mathrm{i}\frac{\partial}{\partial\beta}$ and C is given by (2.35).

Denote by C_ν a restriction of C on the one-dimensional subspace $\mathcal{H}_\nu$ spanned by $f^{(2)}_{\nu\tau}$ with fixed ν and τ. Then

$$C_\nu = \frac{(1-z^2)^2}{\dot{z}^2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\dddot{z}}{\dot{z}} - \frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 + \frac{\dot{z}^2(2+z^2)}{4(1-z^2)^2} - \frac{\dot{z}^2\nu^2}{1-z^2}\right]. \tag{2.36}$$

Before proceeding further, note from (2.36) that C_ν is the Schrödinger type if

$$\frac{(1-z^2)^2}{\dot{z}^2} = 1. \tag{2.37}$$

Equation (2.37) suggests that $z = \tanh x$. If we compute C_ν for this z, it becomes

$$C_\nu = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\nu^2 + \frac{1}{4}}{\cosh^2 x} - \frac{1}{4}.$$

Hence the Pöschl–Teller Hamiltonian

$$H_\nu = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\nu^2 + \frac{1}{4}}{\cosh^2 x}$$

is related to the Casimir operator (2.35) as

$$H_\nu = -\left(C + \tfrac{1}{4}\right)\big|_{\mathcal{H}_\nu}.$$

Let ν^2 and ρ^2 be a linear function of E. We can, without loss of generality, put

$$1 + \nu^2 = a_1 E + b_1 \qquad 1 + \rho^2 = a_2 E + b_2.$$

Then it is not difficult to see that the systems governed by the Hamiltonians

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{b_1 z^2(1-z^2) - \frac{3}{4}(1-z^2) - b_2 z^2 + 1}{R} + \frac{z^4(1-z^2)^2}{R^2}\left(a_1 + \frac{a_1 + a_2(2z^2-1)}{z^2(z^2-1)} - \frac{5\Delta}{4R}\right) \tag{2.38}$$

where $\Delta = (a_1 - a_2)^2$, are related to $SO(2,1)$ in the sense that

$$\left(C_\nu + \rho^2 + \frac{1}{4}\right) = -\frac{(1-z^2)^2}{\dot{z}^2}[H - E]$$

provided

$$\dot{z} = \frac{z(1-z^2)}{\sqrt{R(z)}} \tag{2.39}$$

where $R(z) = a_1 z^4 + (a_2 - a_1)z^2$.

This class of solvable potentials includes as special cases important families of Ginocchio potentials [19]. Indeed, putting

$$a_1 = \frac{1}{\gamma^4} - \frac{1}{\gamma^2} \qquad a_2 = \frac{1}{\gamma^4} \qquad b_2 = 1 \qquad b_1 = \delta(\delta+1) + \frac{3}{4}$$

with $\gamma \leqslant 1$, $\delta(\delta+1) \geqslant \frac{1}{4}$ and introducing Ginocchio's variable y

$$y = \frac{z}{\sqrt{z^2 + \gamma^2(1-z^2)}} \tag{2.40}$$

we have

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \gamma^2\delta(\delta+1)(1-y^2) + \frac{(1-y^2)(1-\gamma^2)}{4}[2 - y^2(7-\gamma^2) + 5(1-\gamma^2)y^4]. \tag{2.41}$$

It follows from (2.39) and (2.40) that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1-y^2)[1-(1-\gamma^2)y^2]. \tag{2.42}$$

We also mention that if $\gamma = 1$ the Hamiltonian in (2.41) simplifies to the Pöschl–Teller Hamiltonian

$$V(x) = \frac{\delta(\delta+1)}{\cosh^2 x}$$

having $SO(2,1)$ as the potential group.

Among others, this algebraic structure provides an integral representation for the wavefunctions. By arguments very similar to those used to obtain (2.30), we have from (A.13), that

$$\Psi_\tau(x) = \frac{(1-z^2)^{\frac{1}{4}}}{\sqrt{\dot{z}}} \int_{-\infty}^{\infty} \left[\frac{\cosh\alpha}{\sqrt{1-z^2}} - \tau\frac{z}{\sqrt{1-z^2}}\right]^{-1-j} \exp(-\mathrm{i}\nu\alpha)\,\mathrm{d}\alpha \tag{2.43}$$

with $j = -\frac{1}{2} - \mathrm{i}\rho$ and $\tau = \pm 1$. Hence we have

$$\begin{aligned}\Psi_\tau(x) \propto{}& (\dot{z})^{-\frac{1}{2}}(1-z^2)^{\frac{2j+3}{4}} \\ &\times \left\{\Gamma\left(\frac{j+\mathrm{i}\nu+1}{2}\right)\Gamma\left(\frac{j-\mathrm{i}\nu+1}{2}\right){}_2F_1\left(\frac{j+\mathrm{i}\nu+1}{2}, \frac{j-\mathrm{i}\nu+1}{2}; \frac{1}{2}; z^2\right)\right. \\ &\left. + 2\tau z\Gamma\left(\frac{j+\mathrm{i}\nu+2}{2}\right)\Gamma\left(\frac{j-\mathrm{i}\nu+2}{2}\right){}_2F_1\left(\frac{j+\mathrm{i}\nu+2}{2}, \frac{j-\mathrm{i}\nu+2}{2}; \frac{3}{2}; z^2\right)\right\}.\end{aligned}$$

It should be noted that the potential functions of this class admit a double degeneracy of the wavefunction for every positive value of energy. (The twofold degeneracy corresponds to the fact that each UIR of $SO(1,1)$ is twofold degenerate in principal series of UIR of $SO(2,1)$.) Therefore, one may construct wave packets which are partly transmitted and partly reflected by the potential. The function Ψ_{-1} represents a wave incident from the left. Reflection occurs at the potential barrier, but there is also transmission to the right. A similar interpretation of Ψ_{+1} can be made. It represents a wave incident from the right, and transmitted through the barrier to the left. According to (1.6), the reflection and transmission coefficients are

$$|R|^2 = \frac{\cosh^2\pi\nu}{\cosh^2\pi\nu + \sinh^2\pi\rho} \qquad |T|^2 = \frac{\sinh^2\pi\rho}{\cosh^2\pi\nu + \sinh^2\pi\rho}.$$

2.3. A class of potentials related to $SO(2,1) \supset E(1)$

We now choose the quasi-invariant measure in (2.8) as $\mathrm{d}\mu = v(\xi_+)\,\mathrm{d}\xi$, with $\xi_+ = \xi_0 + \xi_1$. Such a defined measure is invariant under the transformation given by (2.6). If we calculate the Casimir operator in this realization, we then find

$$\begin{aligned}C ={}& \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} + \left(\frac{\xi_0(\xi_0+\xi_1)}{\xi_2 v}\frac{\partial v}{\partial\xi_2} + 1 + \Lambda\right)\Lambda + \frac{\xi_0}{\xi_2 v}\frac{\partial v}{\partial\xi_2}\frac{\partial}{\partial\xi_1} \\ &+ \frac{\xi_0^2(\xi_0+\xi_1)}{2\xi_2 v}\left[\frac{\partial^2 v}{\partial\xi_1\partial\xi_2} - \frac{(\xi_0+\xi_1)}{2\xi_2 v}\left(\frac{\partial v}{\partial\xi_2}\right)^2 + \frac{(2\xi_0+\xi_1)}{\xi_0^2}\frac{\partial v}{\partial\xi_2}\right]\end{aligned}$$

where we have used

$$\frac{\partial v}{\partial \xi_1} = \frac{\xi_+}{\xi_2}\frac{\partial v}{\partial \xi_2}.$$

We want to diagonalize an infinitesimal operator N corresponding to (2.6)

$$N = \mathrm{i}\left(\xi_2 \frac{\partial}{\partial \xi_1} - \xi_+ \frac{\partial}{\partial \xi_2}\right). \tag{2.44}$$

Hence, the new variables x, β with $-\infty < x < \infty, -\infty < \beta < \infty$ are introduced in this way

$$\xi_1 = \frac{1 - z^2(x) - \beta^2}{2z(x)} \qquad \xi_2 = \frac{\beta}{z(x)} \tag{2.45}$$

where z is a differentiable function on R with values in R^+. Then $N = -\mathrm{i}\frac{\partial}{\partial \beta}$ and

$$C = \frac{z^2}{\dot{z}^2}\left\{\frac{\partial^2}{\partial x^2} + \left(\frac{\dot{v}}{v} - \frac{\ddot{z}}{\dot{z}}\right)\frac{\partial}{\partial x} + \dot{z}^2\frac{\partial^2}{\partial \beta^2} + \frac{1}{2v}\left[\ddot{v} - \frac{\dot{v}^2}{2v} - \frac{\ddot{z}}{\dot{z}}\dot{v}\right]\right\}. \tag{2.46}$$

Since we want the first derivative to vanish, we require

$$\frac{\dot{v}}{v} - \frac{\ddot{z}}{\dot{z}} = 0.$$

We can, without loss of generality, put

$$v = -\dot{z} \quad \text{with} \quad \dot{z} < 0. \tag{2.47}$$

(Since v must be positive we require $\dot{z} < 0$.) Substituting equation (2.47) into equation (2.46), one gets

$$C = \frac{z^2}{\dot{z}^2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\dddot{z}}{\dot{z}} - \frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 + \dot{z}^2\frac{\partial^2}{\partial \beta^2}\right].$$

The restriction of C to the subspace $\mathcal{H}_\lambda$ spanned by $f_\lambda^{(3)}$ for a given λ yields the differential operator C_λ

$$C_\lambda = \frac{z^2}{\dot{z}^2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\dddot{z}}{\dot{z}} - \frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 - \lambda^2\dot{z}^2\right]. \tag{2.48}$$

Then it is not difficult to see that a class of restricted confluent Natanzon potentials [17]

$$V(x) = \frac{g_2 z^2 + h_0 + 1}{R} + \frac{z^2}{R^2}\left(-\sigma_2 + \frac{5\sigma_2 c_0}{R}\right) \tag{2.49}$$

which are defined in terms of four parameters h_0, c_0, g_2, σ_2 and a function $z(x)$ satisfying

$$\dot{z} = -\frac{2z}{\sqrt{R}}$$

where $R(z) = \sigma_2 z^2 + c_0$ are related to C_λ as

$$C_\lambda + \rho^2 + \frac{1}{4} = -\frac{z^2}{\dot{z}^2}(H - E) \qquad H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{2.50}$$

provided

$$4\lambda^2 = g_2 - \sigma_2 E \qquad 1 + 4\rho^2 = c_0 E - h_0.$$

(In (2.49) we closely follow the notation of [17].)

Moreover it follows from (A.17) that

$$\Psi(x) \propto (-\dot{z})^{1/2}(z)^{1+j}\int_{-\infty}^{\infty}[z^2 + t^2]^{-1-j}\,\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \tag{2.51}$$

where $j = -\frac{1}{2} - \mathrm{i}\rho$. Hence for the wavefunctions of restricted confluent Natanzon potentials (2.49) we have

$$\Psi \propto \left(-\frac{z}{\dot{z}}\right)^{1/2} K_{\frac{1}{2}+j}(|\lambda|z) \tag{2.52}$$

where K_ν are the modified Bessel functions of the third kind.

A simple case comes about by choosing $\sigma_2 = 0$, $c_0 = 4$ and $h_0 = -1$. In this case $z(x) = \mathrm{e}^{-x}$, and the potential in (2.49) simplifies to the Toda potential [14]

$$V(x) = \frac{g_2}{4}\,\mathrm{e}^{-2x}. \tag{2.53}$$

3. Conclusions

In this paper we have investigated the scattering systems described by Hamiltonians for which relation (1.8) holds. With such Hamiltonians the theory of intertwining operators allows an explicit algebraic determination of the scattering matrix. Moreover, once the Casimir operator of $SO(2,1)$ is given, equation (1.8) determines in general a family of Hamiltonians whose scattering eigenstates are associated with the basis functions of the carrier space of the principal series of $SO(2,1)$.

According to three subgroup reductions $SO(2,1) \supset SO(2)$, $SO(2,1) \supset SO(1,1)$ and $SO(2,1) \supset E(1)$ provided by the representation theory we have three classes of one-dimensional scattering problems related to $SO(2,1)$ in the sense of equation (1.8). Each class includes a certain set of Natanzon potentials. It is a characteristic difference between the classes that the *S*-matrix for the first and third classes (related to $SO(2,1) \supset SO(2)$ and $SO(2,1) \supset E(1)$ reductions, respectively) is a complex number of unit modules, and for the second class (related to $SO(2,1) \supset SO(1,1)$ reduction) it is a unitary 2×2 matrix. The basic reason for this is that in the principal series of $SO(2,1)$ the spectra of $SO(2)$ and $E(1)$ generators are simple, while the $SO(1,1)$ generator has the multiplicity 2. We should also note that all potentials in (2.27), (2.38) and (2.49) are repulsive and do not support bound states. This is because only the principal series representations appear in the decomposition of the quasiregular representation realized in the space of functions on the upper sheet of hyperboloid $SO(2,1)/SO(2)$.

Acknowledgment

The authors would like to thank S A Baran for discussions.

Appendix. The basis functions of the principal series representations induced by (2.10)

In this appendix we will give the integral representation for the basis functions of the principal series representations induced by (2.10). The procedure is as follows. We shall start with the principal series representations of $SO(2,1)$, induced by the minimal parabolic subgroup [16] of $SO(2,1)$. For such a realization of the principal series the basis functions have a particularly simple form. Then the interrelation between two alternative realizations of the principal series allows us to obtain the integral representation mentioned above.

The principal series representations U^ρ of $SO(2,1)$ labelled by ρ, $0 \leqslant \rho < \infty$ can be realized in the space of infinitely differentiable functions $F(\zeta)$ on the upper sheet of the two-dimensional cone $\zeta_0^2 - \zeta_1^2 - \zeta_2^2 = 0$, $\zeta_0 > 0$, homogeneous of degree $j = -\frac{1}{2} - \mathrm{i}\rho$

$$F(a\zeta) = a^j F(\zeta) \qquad a > 0. \tag{A.1}$$

The representation U^ρ is defined by

$$U^\rho(g)F(\zeta) = F(\zeta g). \tag{A.2}$$

It is worth mentioning that the homogeneous functions on the cone are uniquely determined by their values on any contour Γ intersecting each generator at one point. Hence, U^ρ can be realized in spaces of functions on these contours (see appendix of [15]). The interrelation between this representation and the principal series representation induced by (2.10) is given by the integral transform [20]

$$\check{f}(\xi) = \int_\Gamma [\xi, n]^{-1-j} F(n)\,\mathrm{d}n \equiv (IF)(\xi) \tag{A.3}$$

where $[\cdot,\cdot]$ is given by (2.1) and Γ is an arbitrary contour on the cone which intersects every generator once; and $\mathrm{d}n$ is a quasi-invariant measure on Γ. Moreover the following intertwining relation is held

$$IU = \check{T}I. \tag{A.4}$$

Thus, equation (A.3) allows us to obtain the integral representation for the basis functions of the principal series representations induced by (2.10).

1. $\Gamma = \Gamma_S$, where Γ_S is the section of the cone by plane $\zeta_0 = 1$. Let us introduce the spherical coordinate systems on the cone

$$\zeta = \omega n \qquad n = (1, \cos\theta, \sin\theta) \tag{A.5}$$

where $0 \leqslant \omega < \infty$, $0 \leqslant \theta < 2\pi$. It follows from (A.1) that the function $F(\zeta)$ is uniquely determined by its values on the circle $\Gamma_S = \{n = (1, \cos\theta, \sin\theta) | 0 \leqslant \theta < 2\pi\}$.

$$U^\rho(g)F(n) = (\omega_g)^j F(n_g) \tag{A.6}$$

where ω_g and n_g are determined from parametrization (A.5) of ng, i.e. from $ng = \omega_g n_g$. The Casimir operator of the representation (A.2) is identically a multiple of the unit $C \equiv -\rho^2 - \frac{1}{4}$. The relevant basis on the carrier space of representation (A.6) is given by the reduction $SO(2, 1) \supset SO(2)$. Since $J_0 = -\mathrm{i}\frac{\partial}{\partial\theta}$, the corresponding basis functions are $F_m^{(1)}(n) = \mathrm{e}^{\mathrm{i}m\theta}$. Then, due to (A.3), we come to the following integral representation for the basis functions of the principal series representations induced by (2.10)

$$\check{f}_m^{(1)}(\xi) = \int_0^{2\pi} (\xi_0 - \xi_1\cos\theta - \xi_2\sin\theta)^{-1-j}\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{d}\theta \tag{A.7}$$

where the upper index 1 refers to the $SO(2, 1) \supset SO(2)$ reduction. It is not difficult to see that the basis functions $\check{f}_m^{(1)}$ are indeed the eigenfunctions of the set of commuting operators $\check{C}$ and $\check{J}_0$

$$\check{C}\check{f}_m^{(1)} = j(j+1)\check{f}_m^{(1)} \qquad j = -\tfrac{1}{2} - \mathrm{i}\rho \tag{A.8}$$

$$\check{J}_0\check{f}_m^{(1)} = m\check{f}_m^{(1)} \tag{A.9}$$

where $\check{C}$ is the Casimir operator of $\check{T}$, while $\check{J}_0$ is its infinitesimal operator corresponding to $SO(2)$

$$\check{C} = \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} + (\Lambda+1)\Lambda \qquad \Lambda = \xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2} \tag{A.10}$$

$$\check{J}_0 = \mathrm{i}\left(\xi_2\frac{\partial}{\partial\xi_1} - \xi_1\frac{\partial}{\partial\xi_2}\right). \tag{A.11}$$

2. $\Gamma = \Gamma_H$, where Γ_H are the sections of the cone by planes $\zeta_2 = \pm 1$. According to this, we introduce the hyperbolic coordinate system on the cone

$$\zeta = \omega n_\tau \qquad n_\tau = (\cosh\alpha, \sinh\alpha, \tau) \tag{A.12}$$

where $0 \leqslant \omega < \infty$, $-\infty < \alpha < \infty$ and $\tau \equiv \operatorname{sign}\zeta_2$. By arguments very similar to those used to obtain (A.7) we can show that

$$\check{f}^{(2)}_{\nu\tau}(\xi) = \int_{-\infty}^{\infty} (\xi_0\cosh\alpha - \xi_1\sinh\alpha - \tau\xi_2)^{-1-j}\,\mathrm{e}^{-\mathrm{i}\nu\alpha}\,\mathrm{d}\alpha \tag{A.13}$$

where the upper index 2 refers to the $SO(2,1) \supset SO(1,1)$ reduction. Furthermore,

$$\check{C}\check{f}^{(2)}_{\nu\tau} = j(j+1)\check{f}^{(2)}_{\nu\tau} \tag{A.14}$$

$$\check{J}_1\check{f}^{(2)}_{\nu\tau} = \nu\check{f}^{(2)}_{\nu\tau} \tag{A.15}$$

where $\check{J}_1 = \mathrm{i}\xi_0\frac{\partial}{\partial\xi_1}$ is the infinitesimal operator of $\check{T}$ corresponding to the pure Lorentz transformation $g_1(t)$.

3. $\Gamma = \Gamma_P$, where Γ_P is the section of the cone by the plane $\zeta_0 + \zeta_1 = 1$. We now introduce the parabolic (or horispherical) coordinate system on the cone

$$\zeta = \omega n \qquad n = \left(\frac{1+t^2}{2}, \frac{1-t^2}{2}, t\right) \tag{A.16}$$

where $0 \leqslant \omega < \infty$, $-\infty < t < \infty$. Then it follows from (A.3) that

$$\check{f}^{(3)}_{\lambda}(\xi) = \int_{-\infty}^{\infty} \left(\frac{1+t^2}{2}\xi_0 - \frac{1-t^2}{2}\xi_1 - t\xi_2\right)^{-1-j} \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \tag{A.17}$$

where n is given by (A.16) and the upper index 3 refers to the $SO(2,1) \supset E(1)$ reduction. We note that

$$\check{C}\check{f}^{(3)}_{\lambda} = j(j+1)\check{f}^{(3)}_{\lambda} \tag{A.18}$$

$$\check{N}\check{f}^{(3)}_{\lambda} = \lambda\check{f}^{(3)}_{\lambda} \tag{A.19}$$

where $\check{N} = \mathrm{i}\left(\xi_2\frac{\partial}{\partial\xi_1} - \xi_+\frac{\partial}{\partial\xi_2}\right)$, $\xi_+ = \xi_0 + \xi_1$.

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